

Hardness of almost embedding simplicial complexes in \mathbb{R}^d

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Abstract

A map $f: K \rightarrow \mathbb{R}^d$ of a simplicial complex is an *almost embedding* if $f(\sigma) \cap f(\tau) = \emptyset$ whenever σ, τ are disjoint simplices of K .

Theorem. *Fix integers $d, k \geq 2$ such that $d = \frac{3k}{2} + 1$.*

(a) *Assume that $P \neq NP$. Then there exists a finite k -dimensional complex K that does not admit an almost embedding in \mathbb{R}^d but for which there exists an equivariant map $\tilde{K} \rightarrow S^{d-1}$.*

(b) *The algorithmic problem of recognition almost embeddability of finite k -dimensional complexes in \mathbb{R}^d is NP hard.*

The proof is based on the technique from the Matoušek-Tancer-Wagner paper (proving an analogous result for embeddings), and on singular versions of the higher-dimensional Borromean rings lemma and a generalized van Kampen–Flores theorem.

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1 Introduction

In this paper we study almost embeddings and equivariant maps of configuration spaces. They appear in studies of embeddings [FKT94, Sko08] as well as in topological combinatorics (for Tverberg-type problems see [BZ16, BBZ16, Sko16]). Almost embeddings also turned out to be a useful tool for studying Helly-type results on convex sets, implicitly in [Mat97] and explicitly in [GPP⁺15]. See definitions and more motivations below.

Throughout this paper, let K be a finite simplicial complex.

A map $f: |K| \rightarrow \mathbb{R}^d$ is an **almost embedding** if $f(\sigma) \cap f(\tau) = \emptyset$ whenever σ, τ are disjoint simplices of K . (Existence of an almost embedding is obviously a necessary condition for existence of an embedding.)

The **(simplicial) deleted product** of K is

$$\tilde{K} := \cup \{ \sigma \times \tau : \sigma, \tau \text{ are simplices of } K, \sigma \cap \tau = \emptyset \};$$

i.e., \tilde{K} is the union of products $\sigma \times \tau$ formed by disjoint simplices of K .

Suppose that $f: |K| \rightarrow \mathbb{R}^d$ is an almost embedding. Then the map $\tilde{f}: \tilde{K} \rightarrow S^{d-1}$ is well-defined by the Gauss formula

$$\tilde{f}(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}.$$

We have $\tilde{f}(y, x) = -\tilde{f}(x, y)$; i.e., this map is **equivariant** with respect to the ‘exchanging factors’ involution $(x, y) \rightarrow (y, x)$ on \tilde{K} and the antipodal involution on S^{d-1} . Thus the existence of an equivariant map $\tilde{K} \rightarrow S^{d-1}$ is a necessary condition for almost embeddability of $|K|$ in \mathbb{R}^d .

Theorem 1. *Fix each integers $d, k \geq 2$ such that $d = \frac{3k}{2} + 1$.*

- (a) *Assume that $P \neq NP$. Then there exists a finite k -dimensional complex K that does not admit an almost embedding in \mathbb{R}^d but for which there exists an equivariant map $\tilde{K} \rightarrow S^{d-1}$.*
- (b) *The algorithmic problem of recognition almost embeddability of finite k -dimensional complexes in \mathbb{R}^d is NP hard.*

The reader need not to know what NP hardness is: the essence of part (b) is explained by Theorem 2 below.

For $k = 2$ part (a) is true even without $P \neq NP$ assumption, by [AMSW16, Theorem 1.5 and Proposition 1.7]. Part (a) follows by part (b) and the existence of a polynomial algorithm for checking the existence of equivariant maps [ČKV13]. Indeed, for fixed d, k it is polynomial time decidable whether there exists an equivariant map $\tilde{K} \rightarrow S^{d-1}$ [ČKV13]. Given that almost embeddability implies the existence of an equivariant map, part (b) implies part (a).

We discuss the possibility of removing the assumption $P \neq NP$ at the end of the introduction.

Remark. The conclusions of Theorem 1 are in fact valid for each fixed integers k, d such that $2 \leq k \leq d \leq \frac{3k}{2} + 1$ and $d \equiv 1 \pmod{3}$. That is, we reflect only the interesting extremal cases in the statement of Theorem 1.

Indeed, for such integers k, d let $k' := \frac{2(d-1)}{3}$. For a proof of part (a) with (d, k) we take the complex K' from part (a) with parameters (d, k') , and define the complex K to be the disjoint union of K' and a k -simplex. Similarly, for a proof of part (b) with (d, k) we add an isolated k -simplex to every k' -complex. In both cases it is easy to check that the conclusion of Theorem 1 remains valid as for $d \geq k$ adding an isolated k -simplex does not affect neither almost embeddability to \mathbb{R}^d nor the existence of an equivariant map to S^{d-1} .

Motivation and background. A classical question in topology is to determine whether a simplicial complex K embeds (topologically/piecewise linearly/linearly) in \mathbb{R}^d . It is easy to deduce that every k -dimensional simplicial complex embeds (even linearly) into \mathbb{R}^{2k+1} . Pioneering result in this area, known as the van Kampen–Flores theorem [vK33, Flo34, Sko14], states the existence of k -dimensional complexes that do not embed into \mathbb{R}^{2k} (for every integer k ; even topologically).

In general, it is often very hard to determine whether a given complex K embeds into \mathbb{R}^d . More precisely, this question subtly depends on the comparison of $k := \dim K$ and d . For example, it is algorithmically undecidable to recognize whether a given $(d-1)$ -complex embeds into \mathbb{R}^d [MTW11], provided that $d \geq 5$. (This result follows from a celebrated theorem of Novikov on unrecognizability of the d -sphere. [VKF74].)

Matoušek, the second author and Wagner [MTW11] proved that for each pair (k, d) such that $4 \leq d \leq \frac{3n}{2} + 1$ it is NP hard to decide whether a k -dimensional simplicial complex PL embeds in \mathbb{R}^d . Theorem 1(b) is a version of this result for almost embeddability.

We describe the method from [MTW11] in detail (in order to prove our main results), up to one step in proof that we take directly from [MTW11]. We ex-

licitly state the initial step of the proof (Theorem 2 below). Next, we slightly simplify the main construction (construction of $K(\Phi)$ in §2). We also present a simple proof of Lemma 7 below generalizing the van Kampen-Flores theorem (which is also one of the key tools for the result). For $\ell = k - 1$ this lemma is proved in [vK33], for $\ell < k - 1$ a weaker version of this lemma (when $f|_{S_1}$ is a PL embedding) is proved in [SS92, Lemma 1.4] using the Smith index. Thus this paper can serve as an exposition of the proof of the above result of [MTW11].

Theorem 1(b) is interesting on its own because almost embeddability is different from embeddability. Consider the following three properties of a finite simplicial complex K .

(E) K PL embeds into \mathbb{R}^d .

(AE) K PL almost embeds in \mathbb{R}^d .

(EM) There exists an equivariant map $\tilde{K} \rightarrow S^{d-1}$.

The conditions (AE) and (EM) appeared as ‘combinatorial’ or ‘algebraic’ counterparts of (E), useful to study ‘geometric’ condition (E). Theorem 1 indicates that the condition (AE) is closer to (E) than to (EM), from algorithmic point of view. More precisely, we have

$$\begin{array}{c} E \implies AE \implies EM \\ \curvearrowright \\ 2d \geq 3k+3 \text{ or } d \leq 2 \end{array}$$

Here the straight arrows are clear and explained above, and the curved arrow is a theorem of Weber [Web67]; see also [Sko08, §5]. For every pair (k, d) such that ‘ $2d \geq 3k + 3$ or $d \leq 2$ ’ does not hold, i.e. such that $3 \leq d \leq \frac{3k}{2} + 1$, we have $(EM) \not\Rightarrow (E)$ [SS92], [FKT94], [SSS98], [GS06]. Moreover, for every pair (k, d) such that $4 \leq d \leq \frac{3k}{2} + 1$ we have $(AE) \not\Rightarrow (E)$ [SS92], [SSS98]. See [Sko08, §5, §7] for a survey. By [AMSW16, Theorem 1.5 and Proposition 1.7] $(EM) \not\Rightarrow (AE)$ for $d = 2k = 4$. Theorem 1(a) shows (modulo $P \neq NP$) that for every pair (k, d) such that $4 \leq d = \frac{3k}{2} + 1$ $(EM) \not\Rightarrow (AE)$.

We prove Theorem 1 by applying the technique of [MTW11] (which builds on a construction in [SS92], [FKT94], [SSS98], see [Sko08, §5, §7] for a survey), the Singular Borromean Rings Lemma 4 [AMSW16] and Lemma 7. Formally, Theorem 1 follows by NP-hardness of recognition of 3-SAT problem and the following result.

A **3-CNF formula** in variables x_1, \dots, x_n is

$$\Phi(x_1, \dots, x_n) = \bigwedge_{s=1}^t (x_{n_{s1}}^{\alpha_{s1}} \vee x_{n_{s2}}^{\alpha_{s2}} \vee x_{n_{s3}}^{\alpha_{s3}}).$$

Here $n_{si} \in \{1, \dots, n\}$, $\alpha_{si} \in \{0, 1\}$ and $x^0 = \neg x$, $x^1 = x$.

Theorem 2. *Let $d, k \geq 2$ be fixed integers such that $d = \frac{3k}{2} + 1$. Then to each 3-CNF formula Φ there corresponds, by a polynomial algorithm (in the size of the formula Φ), a finite k -dimensional complex $K(\Phi)$ such that $K(\Phi)$ is almost embeddable in \mathbb{R}^d if and only if Φ is satisfiable (i.e., if the Boolean function $\mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$ corresponding to Φ is not identically zero).*

The analogue of Theorem 2 for embeddability is proved in [MTW11, §4, §5]. Our complex $K(\Phi)$ slightly differs from the complex constructed in [MTW11, §4.2, §5.2], which we call $K'(\Phi)$.¹ For the ‘if’ part of Theorem 2 we need the following lemma.

Lemma 3 (proof is sketched in Section 2). *The complex $K(\Phi)$ constructed in Section 2 is obtained from the complex $K'(\Phi)$ constructed in [MTW11, §4.2, §5.2] by several contractions of edges and several compressions of $S^{\ell-1} \times I$. (Compression of a subcomplex identified with $S^{\ell-1} \times I$ is contracting to a point each segment $x \times I$, $x \in S^{\ell-1}$.)*

Proof of Theorem 2: the ‘if’ part. Since the formula Φ is satisfiable, the complex $K'(\Phi)$ PL embeds into \mathbb{R}^d [MTW11, Section 5].² Recall that the quotient of a PL manifold by a map with collapsible point-inverses is PL homeomorphic to the same manifold [Coh67]. Hence contracting an edge and compressing $S^{\ell-1} \times I$ keep PL embeddability.³ Thus by Lemma 3 $K(\Phi)$ also PL embeds into \mathbb{R}^d . \square

Therefore it suffices to prove the converse: if $K(\Phi)$ almost-embeds in \mathbb{R}^d , then Φ is satisfiable. This is a strengthening of the analogous fact from [MTW11]. We use the same idea as [MTW11] but need to replace two key lemmas [MTW11, Lemmas 5.1(i) and 5.3(i)] by suitable analogues for almost embeddings. These analogues are Lemma 5 and the Singular Borromean Rings Lemma 4 [AMSW16, Lemma 1.9] below, respectively.

Let $T := S^\ell \times S^\ell$ be the 2ℓ -dimensional torus with meridian $a := S^\ell \times \cdot$ and parallel $b := \cdot \times S^\ell$. See well-known definition of ‘linked modulo 2’, e.g., in [ST80, §77] or in [Sko, §2.2 ‘Linking modulo 2’].

Lemma 4 (Singular Borromean Rings). *For each $k = 2\ell$ let S_a^k and S_b^k be copies of S^k . Then there is no PL map $f: T \sqcup S_a^k \sqcup S_b^k \rightarrow \mathbb{R}^{k+\ell+1}$ such that*

¹We make this minor change to simplify the construction of $K(\Phi)$ and proof of the ‘only if’ part, which, however, would work for $K'(\Phi)$ as well.

²The converse is also true but is not used here.

³The analogous claim is not true for ‘decontractions’.

- (a) the f -images of the components are pairwise disjoint;
- (b) $f(S_a^k)$ is linked modulo 2 with $f(a)$ and is not linked modulo 2 with $f(b)$;
- (c) $f(S_b^k)$ is linked modulo 2 with $f(b)$ and is not linked modulo 2 with $f(a)$.

On the assumption $P \neq NP$. A reader could expect that analyzing the algorithm in [ČKV13] and the proof of Theorem 1(b) would yield a direct construction of an example of Theorem 1(a), without the assumption $P \neq NP$. Here we discuss the difficulties that appear in this analysis.

Let us consider a 3-CNF formula Φ and let d and k be fixed, $d = \frac{3k}{2} + 1$. If Φ is satisfiable, then (AE) holds for $K(\Phi)$ by Theorem 2 and therefore (EM) holds for $K(\Phi)$ as well.

If Φ is not satisfiable, then (AE) does not hold for $K(\Phi)$ by Theorem 2 but we do not know whether (EM) holds for $K(\Phi)$. However, if (EM) did not hold for every Φ which is not satisfiable, then we would deduce that it is NP-hard to recognize whether a given simplicial complex K satisfies (EM) . On the other hand, this is a polynomial time solvable problem due to [ČKV13]. This would be only possible if $P = NP$.

Thus, we have good reasons to expect that for any choice of k and d with $d = \frac{3k}{2} + 1$, there is a non-satisfiable formula Φ such that (EM) holds for $K(\Phi)$. Actually, we conjecture that (EM) holds for $K(\Phi)$ for every Φ .

For a proof of Theorem 1(a), without the assumption $P \neq NP$, it would be fully sufficient to exhibit a single non-satisfiable 3-CNF formula Φ such that (EM) holds for $K(\Phi)$. In fact, the construction in Theorem 2 makes also sense for the simplest non-satisfiable 1-CNF formula $\Phi_{\text{NO}} = x_1 \wedge \neg x_1$ (in the definition of the clause gadget G in Section 2 only a single simplex is removed instead of three simplices). Let $K_{\text{NO}}(k, d) := K(\Phi_{\text{NO}})$ for given parameters d and k (here we want to emphasize the dependence on k and d). The complex $K_{\text{NO}}(2, 4)$ is essentially the complex constructed by Freedman, Krushkal and Teichner [FKT94] (up to a minor modification), and we know that (EM) holds in this case (as we discussed below the statement of Theorem 1).

For few other values of k and d we could, in principle, run the algorithm of [ČKV13] on $K_{\text{NO}}(k, d)$ (unfortunately, it is not implemented⁴). However, we do not know how to verify (EM) for infinitely many values of k and d we are interested in; the dependence of the algorithm from [ČKV13] on k and d is somewhat complicated. The algorithm in [ČKV13] is based on the obstruction theory. As far as we know there are no other tools developed, besides the obstruction theory, that would allow us to verify (EM) for our examples ‘by hand’.

⁴That is, the code is not written.

2 Proof of Theorem 2

First we define building blocks for the complex $K(\Phi)$ (most importantly, clause gadgets) and prove their properties.

Take any integers $0 \leq \ell < k$. We suppress dependence on k, ℓ . For an integer n denote

$$[n] := \{1, \dots, n\}.$$

Definition of an auxiliary complex F . Complex F has the vertex set $[k + \ell + 3] \cup \{p\}$. The simplices are

- complete k -skeleton on $[k + \ell + 3]$, and
- all the simplices of dimension at most $\ell + 1$ that contain p .

In other words,

$$F := \left([k + \ell + 3] \cup \{p\}, \left(\begin{matrix} [k + \ell + 3] \\ \leq k + 1 \end{matrix} \right) \cup \left\{ \{p\} \cup \sigma : \sigma \in \left(\begin{matrix} [k + \ell + 3] \\ \leq \ell + 1 \end{matrix} \right) \right\} \right).$$

Here $\binom{n}{\leq m}$ is the set of all subsets of $[n]$ having at most m elements.

Definitions of σ_j , S_j and clause gadget G . In this definition j is any element of $[3]$.

Set σ_j to be the simplex with vertex set $\{p\} \cup [\ell + 2] - \{j\}$. Then σ_1, σ_2 and σ_3 are three $(\ell + 1)$ -simplices containing p .

Set S_j to be the union of all k -simplices with vertices in $[k + \ell + 3]$ that do not intersect σ_j . Clearly, this union is homeomorphic to the k -sphere.⁵

Finally, we define the *clause gadget* as $G := F - \sigma_1 - \sigma_2 - \sigma_3$.

Lemma 5. *For any $\ell < k$ and general position PL almost-embedding $f : |G| \rightarrow \mathbb{R}^{k+\ell+1}$ there is $i \in [3]$ such that $f(\partial\sigma_i)$ is linked modulo 2 with $f(S_i)$.*

This lemma is a stronger ‘singular version’ of [SS92, Lemma 1.4], [FKT94, Lemma 6], [SSS98, Lemma 1.1], [Sko08, Lemma 7.2], and [MTW11, Lemmas 4.1 and 5.1(i)]. The difference is that $f|_{S_j}$ is not assumed to be an embedding. In spite of Lemma 5 being stronger, our proof (presented in Section 3) is simpler. For $k = 2\ell = 2$ Lemma 5 is proved in [FKT94, proof of Lemma 8]. Cf. [AMSW16, Remark 2.3.b].

⁵This was called a *complementary* sphere in [MTW11].

Proof of Theorem 2: construction of $K(\Phi)$. Recall the notation for 3-CNF formula Φ given before Theorem 2. The ‘multiple’ $x_{n_{s1}}^{\alpha_{s1}} \vee x_{n_{s2}}^{\alpha_{s2}} \vee x_{n_{s3}}^{\alpha_{s3}}$ is the s -th clause of Φ . The ‘summand’ $x_{n_{si}}^{\alpha_{si}}$ is the i -th literal of the s -th clause. Without loss of generality, we may assume that no clause (‘multiple’) contains both x_m and $\neg x_m$ for some m (otherwise such a clause would be redundant). Denote

$$P := \{(q, r) \in ([t] \times [3])^2 : n_q = n_r, \alpha_q = 0 \text{ and } \alpha_r = 1\}.$$

This is the set of all pairs $(q, r) = ((q_1, q_2), (r_1, r_2))$ such that for some m

- the q_2 -th literal (‘summand’) of the q_1 -th clause (‘multiple’) is x_m , and
- the r_2 -th literal (‘summand’) of the r_1 -th clause (‘multiple’) is $\neg x_m$.

(In other words, this is the set of all pairs of (pairs of) indices of literals in conflict.)

Take copies G_1, \dots, G_t of (the clause gadget) G . Denote by $\sigma_q = \sigma_{(q_1, q_2)}$ the simplex σ_{q_2} in the copy G_{q_1} . Take a triangulation of k -torus T extending triangulations of its meridian and parallel a and b as boundaries of $(\ell + 1)$ -simplices. For each $(q, r) \in P$ take a copy $T_{qr} \supset a_{qr}, b_{qr}$ of $T \supset a, b$. Set

$$K(\Phi) := \bigcup_{s=1}^t G_s \bigcup_{\substack{\partial\sigma_q=a_{qr}, \partial\sigma_r=b_{qr}, (q,r) \in P}} \cup_{(q,r) \in P} T_{qr}.$$

That is, this complex is obtained from the copies G_s and T_{qr} , by identifying the ℓ -spheres $\partial\sigma_q$ and a_{qr} , and the ℓ -spheres $\partial\sigma_r$ and b_{qr} , for each $(q, r) \in P$.

Recall that k and ℓ are fixed. Then each of the complexes G and T can be built by a constant-time algorithm. Hence $K(\Phi)$ is obtained from Φ by a polynomial algorithm in n and t (i.e. in the size of the formula). \square

Sketch of a proof of Lemma 3. The construction of $K'(\Phi)$ [MTW11, §4.2, §5.2] is different from the above construction of $K(\Phi)$ by the following details:

- the torus T is replaced by a polyhedron X containing ‘parallel’ and ‘meridian’ a and b , and an edge whose contraction yields T ;
- the simplices σ_i in the definitions of G and of $K(\Phi)$ are replaced by k -disks $\omega_i \subset \text{Int } \sigma_i$.

Thus $K(\Phi)$ is obtained from $K'(\Phi)$ by contracting the edge in each copy of T , and by compressing $\sigma_j - \text{Int } \omega_j \cong S^{\ell-1}$ in each copy of G and for each $j \in [3]$. \square

Proof of Theorem 2: the ‘only if’ part. Assume that there is a PL almost embedding $f : |K(\Phi)| \rightarrow \mathbb{R}^{k+\ell+1}$. By Lemma 5 for every $s \in [t]$ there is $i(s) \in [3]$ such that the f -images of the spheres $S_{si(s)}, \partial\sigma_{si(s)} \subset |G_s|$ are linked modulo 2.

Let us assume, for contradiction, that Φ is not satisfiable; that is, $\Phi \equiv 0$. (The reader not so familiar with literals, clauses and conflicts, may wish to skip the next paragraph and check rather the footnote in the following one.)

The function i selects one literal ('summand') in each clause ('multiple'). Then two selected literals ('summands') must be in conflict.

This means that there are $q_1, r_1 \in [t]$ and $m \in [n]$ such that the $i(q_1)$ -th literal ('summand') of the q_1 -th clause ('multiple') is x_m and the $i(r_1)$ -th literal ('summand') of the r_1 -th clause ('multiple') is $\neg x_m$.⁶ That is, $(q, r) := ((q_1, i(q_1)), (r_1, i(r_1))) \in P$.

Then $\partial\sigma_q = a_{qr}$ and $\partial\sigma_r = b_{qr}$. Since f is an almost embedding, the f -images of S_q , S_r and T_{qr} are pairwise disjoint. The sphere S_q bounds the disk $p_{q_1} * S_q$ outside b_{qr} , where p_{q_1} is the distinguished vertex 'p' of G_{q_1} . Hence $f(S_q)$ is unlinked modulo 2 with $f(b_{qr})$. Analogously $f(S_r)$ is unlinked modulo 2 with $f(a_{qr})$. Since $k = 2\ell$, all this contradicts the Singular Borromean Rings Lemma 4 applied to the restriction of f to $S_q \sqcup S_r \sqcup T_{qr}$. \square

3 The van Kampen number: proof of Lemma 5

For a general position PL map $f: |K| \rightarrow \mathbb{R}^d$ of a finite k -complex define *the van Kampen number*

$$v(f) \in \mathbb{Z}_2$$

to be the parity of the number of points $x \in \mathbb{R}^d$ such that $x \in f(\sigma) \cap f(\tau)$ for some disjoint simplices $\sigma, \tau \in K$ with $\dim \sigma + \dim \tau = d$.

Lemma 6. *Let K be a finite complex such that for every pair σ, τ of disjoint s - and t -simplices in K such that $s + t = d - 1$ the sum N of the following two numbers is even:*

- *the number of $(s + 1)$ -simplices ν containing σ and disjoint with τ ;*
- *the number of $(t + 1)$ -simplices μ containing τ and disjoint with σ .*

Then $v(f)$ is independent of a general position PL map $f: |K| \rightarrow \mathbb{R}^d$.

For $d = 2$ and $K = K_5$ this corresponds to well-known proof of the non-planarity of K_5 [Sko, Lemma 3.4], [BE01, §5]. For the general case the proof is analogous.

⁶Indeed, in the opposite case for each $m \in [n]$ there is $\alpha(m) \in \{0, 1\}$ such that $\alpha_{si(s)} = \alpha(n_{si(s)})$ for each $s \in [t]$. So we can take $x_{n_{si(s)}} := \alpha(n_{si(s)})$ for each $s \in [t]$ and then extend this to a satisfying assignment.

Proof of Lemma 6. Lemma 6 follows analogously to [Sha57, Lemma 3.5] (by interpreting $v(f)$ as an obstruction to the existence of certain equivariant map).

A direct proof is as follows. Take a general position PL homotopy $H : |K| \times I \rightarrow \mathbb{R}^d \times I$ between general position PL maps $H_0, H_1 : |K| \rightarrow \mathbb{R}^d$. Then

$$v(H) := \cup \{H(\sigma \times I) \cap H(\tau \times I) : \sigma, \tau \in K, \sigma \cap \tau = \emptyset, \dim \sigma + \dim \tau = d\}$$

is a graph. For each $i = 0, 1$ the vertices of this graph in $\mathbb{R}^d \times i$ are exactly the points x from the definition of $v(H_i)$. So $v(H_i)$ equals to the number modulo 2 of vertices of this graph in $\mathbb{R}^d \times i$. This graph also has vertices in $\mathbb{R}^d \times (0, 1)$ corresponding to pairs (ν, τ) and (σ, μ) from the bullet points of the lemma. (There could be connected components of $v(H)$ containing no such vertices; there could be some other vertices of even degree, e.g. vertices of degree 2 coming from double points of H or vertices of degree 4 coming from triple points of H .)

Analogously to [Hud69, Lemma 11.4] any vertex of this graph

- contained in $\mathbb{R}^d \times i$ has odd degree.
- contained in $\mathbb{R}^d \times (0, 1)$ has even degree (by the bullet points of the lemma).

Hence $v(H_0) = v(H_1)$. \square

Lemma 7. *For any $\ell < k$ and general position PL map $f : |F| \rightarrow \mathbb{R}^{k+\ell+1}$ we have $v(f) = 1 \in \mathbb{Z}_2$.*

Proof. For some f Lemma 7 was proved in [SS92, Lemma 1.1]. Then Lemma 7 follows for any f by Lemma 6 after we verify the assumptions of the lemma.

Take any pair σ, τ of disjoint s - and t -simplices in F such that $s + t = k + \ell$. Without loss of generality we assume that $s \leq t$. Since $t \leq k$, we obtain $s \geq \ell$. Among $(k + 1) + (\ell + 1) + 2$ vertices of F there are exactly two which are not contained in $\sigma \cup \tau$. We distinguish two cases.

- *Case $s = \ell$.* In this case $t = k$. Since $\dim F = k$, the simplex τ cannot be extended to a simplex of F (disjoint with σ). Since F contains complete $(\ell + 1)$ -skeleton, σ can be extended (to $(s + 1)$ -simplex of F disjoint with τ) by both vertices of F not contained in $\sigma \cup \tau$. Thus $N = 0 + 2$ is even.
- *Case $s > \ell$.* In this case $\ell < t < k$.

Subcase when neither σ nor τ contains p . Since p is contained only in simplices of dimension at most $\ell + 1$, neither σ nor τ can be extended by p to a simplex of F . On the other hand, since $s, t < k$, the remaining vertex of F can serve for extension of both σ or τ . Thus $N = 1 + 1$ is even.

Subcase when σ or τ contains p . Since $t > \ell$ and p can be only contained in a simplex of dimension at most $\ell + 1$, we can without loss of generality assume that $p \in \sigma$ and $s = \ell + 1$. Since p does not belong to any $(\ell + 2)$ -simplex, it follows that σ cannot be extended. On the other hand, τ can be extended in two ways to both vertices of F not contained in $\sigma \cup \tau$. Thus $N = 0 + 2$ is even. □

Proof of Lemma 5. Extend the map f to a general position PL map $g: |F| \rightarrow \mathbb{R}^{k+\ell+1}$. Since $\ell + 2k < 2(k + \ell + 1)$, by general position to every point x from the definition of $v(g)$ there corresponds a unique unordered pair of simplices of F , the sum of whose dimensions is d and the intersection of whose g -images contains x . Since f is an almost-embedding, for every such point x there is a unique $i \in [3]$ such that $x \in g(\sigma_i) \cap g(S_i)$. This and Lemma 7 imply that $\sum_{i=1}^3 |g(\sigma_i) \cap g(S_i)| \equiv_2 v(g) = 1 \in \mathbb{Z}_2$. Hence one of the three summands is odd as required. □

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